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On the uniform vorticity in a high Reynolds number flow

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Abstract. A method is proposed for determining the value of the uniform vorticity (ω_0) in the inviscid region of a high Reynolds number (*Re*) flow with closed streamlines. An asymptotic treatment of the area integral of the Navier–Stokes equations over the enclosed region leads to a constraint involving the core vorticity; this requires the solution of the momentum equations at $O(1)$ and $O(Re^{-1/2})$ both in the core and in the surrounding boundary layers, although we are subsequently able to show that, under the assumption that the core vorticity at $O(\delta)$ is also constant, the value of ω_0 depends only on the flow at $O(1)$. The analysis is verified numerically for the case where the boundary is an ellipse, and is also shown to be in agreement with the only case for which an analytic solution is available, namely when the enclosing boundary is circular. The validity of the above-mentioned assumption is also discussed.

1. Introduction

In this paper, we consider the steady, two-dimensional, laminar motion within a closed region which is driven by a prescribed velocity distribution along the external boundary. It is well-known that, if the Reynolds number is large, the resulting motion will be one in which the streamlines are closed and in which there will be an inviscid core of uniform vorticity, separated from the moving boundary by viscous boundary layers [1]; this overall picture has been confirmed by numerical solutions of the Navier–Stokes equations, most frequently for the square cavity ([2, 4, 6, 7, 11, 13, 14]), but occasionally for other geometries [5]. Such solutions are able to provide a numerical value for the uniform vorticity, although the question remains as to whether the core vorticity can be determined without recourse to a numerical solution of the full Navier–Stokes equations, but rather by considering only the boundary-layer equations. This is indeed the case when the boundary is circular [1]; the case of a non-circular boundary was considered by Riley [9], and a criterion for determining the core vorticity was derived in terms of a matching condition at the outer edge of the boundary layer between the inviscid core flow and the boundary-layer flow. The aim of this paper, therefore, is to propose an alternative method for determining the core vorticity for high Reynolds number flow in a closed region.

In Section 2, we use asymptotic methods to derive from the Navier–Stokes equations an integral relation which determines the value of the core vorticity inside a region of arbitrary shape, whilst in Section 3 we show how to calculate the various terms that constitute the relation, namely the velocity fields at $O(1)$ and $O(Re^{-1/2})$. In Section 4 we demonstrate, by means of boundary-layer computations, that our theory produces a result that is consistent with that obtained by Batchelor [1] for the case of a circular boundary; in addition, our computed solutions for a family of ellipses are shown to be consistent with results obtained by Riley [9]. In Section 5, we compare the present paper with earlier work, and draw conclusions.

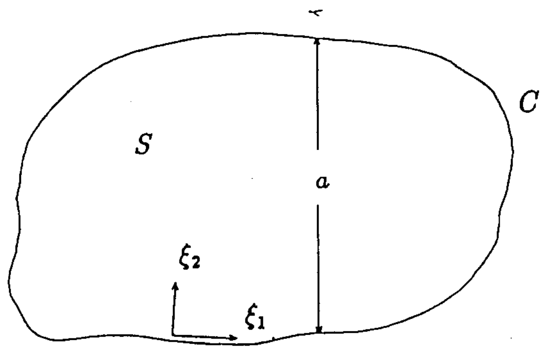

Fig. 1. Geometry for closed-streamline flow.

2. Formulation

Consider a region S surrounded by a closed curve C, as in Fig. 1; C represents a boundary, whose steady motion induces fluid flow within S and across which there is no normal outflow. The tangential motion of the boundary is prescribed and we assume that the Reynolds number (Re) is large; as usual, $Re = \rho U a/\mu$, where ρ is the fluid density, U is a typical velocity scale, a is a typical length scale for the region S and μ is the coefficient of viscosity. The governing flow equations are then

$$\frac{\partial u_i}{\partial x_i} = 0\,, \quad i = 1, 2 \tag{1}$$

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j}\,, \quad i, j = 1, 2 \tag{2}$$

where $(u_i)_{i=1,2}$ are the components of the velocity field, $(x_i)_{i=1,2}$ are Cartesian position coordinates and σ_{ij} is the stress tensor, which is given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}\,,$$

where p is the pressure, δ_{ij} is the Kronecker delta and τ_{ij} is the deviatoric stress tensor, which in turn is given by

$$\tau_{ij} = \frac{1}{Re}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The boundary conditions for the flow are then

$$u_i n_i = 0 \quad \text{on } C\,, \tag{3}$$

where $(n_i)_{i=1,2}$ are the Cartesian components of the outward normal, $\mathbf{n}$, to C, and

$$u_i t_i = U_s \quad \text{on } C \tag{4}$$

where $(t_i)_{i=1,2}$ are the components of the tangent, $\mathbf{t}$, to C and U_s is the prescribed tangential velocity.

Rewriting (2), with the help of (1) as

$$\frac{\partial(u_j u_i)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i, j = 1, 2$$

we obtain, on applying the divergence theorem to S, and using (3),

$$\oint_C \sigma_{ij} n_j \, \mathrm{d}s = \oint_C u_i u_j n_j \, \mathrm{d}s$$
$$= 0 . \tag{5}$$

Since $Re \gg 1$, it is appropriate to adopt an asymptotic approach and, to this end, we write the asymptotic expansions for p and $(u_i)_{i=1,2}$ in the core of S as

$$p = p_0 + \delta p_1 + \cdots ,$$
$$u_i = u_{0i} + \delta u_{1i} + \cdots ,$$

where $\delta^2 = 1/Re$, and consider (5) at leading order. However, the leading order terms are not immediately obvious, given that we are considering p and $((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))$ in the boundary layer, where they are not necessarily $O(1)$ as they would be in the core. Nevertheless, it is well known that $u_i n_i \sim \delta$ and $u_i t_i \sim 1$ in the boundary layer, and that p, to leading order, is simply the inviscid Euler pressure, which satisfies

$$\frac{\partial(u_{0j} u_{0i})}{\partial x_j} = -\frac{\partial p_0}{\partial x_i}, \quad i, j = 1, 2 ,$$

whence, as in (5), we obtain

$$\oint_C p_0 n_i \, \mathrm{d}s = 0 , \quad i = 1, 2 . \tag{6}$$

In order to determine the magnitude of $((\partial u_{0i}/\partial x_j) + (\partial u_{0j}/\partial x_i))$ at C, we write down the asymptotic expansions for (U, V), the tangential and normal components of velocity in the boundary layer, as

$$U = U_0 + \delta U_1 + \cdots ,$$
$$V = \delta V_0 + \delta^2 V_1 + \cdots ,$$

with the pressure p as

$$p = P_0 + \delta P_1 + \cdots .$$

Next, we consider the curvilinear coordinate system (ξ_1, ξ_2), where ξ_1 is tangential to C and ξ_2 is normal to C, so that $\xi_2 = \xi_C$, with ξ_C constant, corresponds to C itself. In the first instance, it is required to show that (5) may be reduced to the pair of equations

$$\oint_C (\sigma_{\xi_2 \xi_2} n_1 + \sigma_{\xi_1 \xi_2} n_2) \, \mathrm{d}s = 0 , \tag{7}$$

$$\oint_C (\sigma_{\xi_2\xi_2} n_2 - \sigma_{\xi_1\xi_2} n_1)\,\mathrm{d}s = 0\,, \tag{8}$$

where n_{ξ_1} and n_{ξ_2} are the direction cosines of the normal to C, and $\sigma_{\xi_1\xi_2}$ and $\sigma_{\xi_2\xi_2}$ are given respectively by

$$\sigma_{\xi_1\xi_2} = \delta^2\left(\frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{h_1}\right) + \frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{h_2}\right)\right),$$

$$\sigma_{\xi_2\xi_2} = -p\delta_{ij} + 2\delta^2\left(\frac{1}{h_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1}\right),$$

where u_1 and u_2 are the first and second components of the velocity field and h_1 and h_2 are the associated scale factors.

With reference to Fig. 2, the scalar product of an arbitrary unit vector **c** with the vector of which the components are $\oint_C \sigma_{ij} n_j\,\mathrm{d}s$ gives

$$\oint_C c_i \sigma_{ij} n_j\,\mathrm{d}s = 0\,.$$

Now, taking components in the (ξ_1, ξ_2)-system, we obtain, since $n_{\xi_1} = 0$ and $n_{\xi_2} = -1$,

$$\oint_C (c_{\xi_1}\sigma_{\xi_1\xi_2} + c_{\xi_2}\sigma_{\xi_2\xi_2})\,\mathrm{d}s = 0\,;$$

next, taking $(c_i)_{i=1,2}$ along the x_1-axis gives

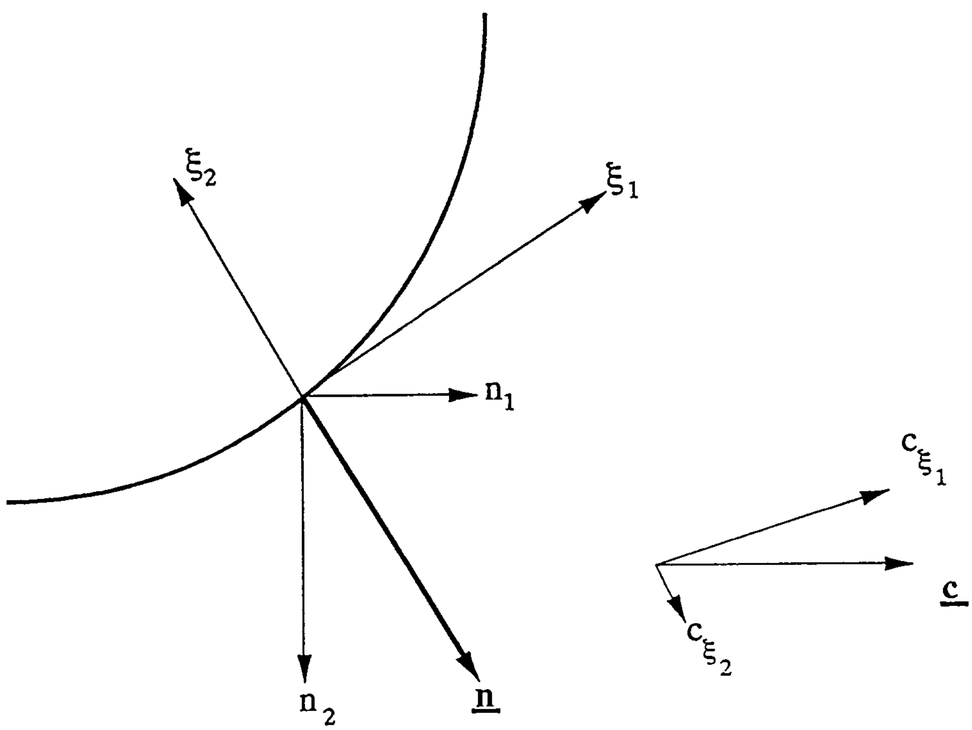

Fig. 2. Plot of curvilinear orthogonal coordinate system.

$$c_{\xi_1} = -n_2 \,, \quad c_{\xi_2} = -n_1 \,,$$

which leads to (7), whilst taking $(c_i)_{i=1,2}$ along the x_2-axis gives

$$c_{\xi_1} = n_1 \,, \quad c_{\xi_2} = -n_2 \,,$$

and thence (8), as required.

Introducing a streamfunction, ψ, given by

$$u_1 = \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2} \,, \quad u_2 = -\frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1} \,,$$

and writing the appropriate asymptotic expansion for ψ as

$$\psi = \psi_0 + \delta\psi_1 + \cdots$$

we proceed by writing $\sigma_{\xi_1\xi_2}$ and $\sigma_{\xi_2\xi_2}$ in terms of ψ, ξ_1 and ξ_2 as

$$\sigma_{\xi_1\xi_2} = \delta^2\Big(\frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\Big(\frac{1}{h_1h_2}\frac{\partial \psi}{\partial \xi_2}\Big) - \frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\Big(\frac{1}{h_1h_2}\frac{\partial \psi}{\partial \xi_1}\Big)\Big) + O(\delta^3) \,,$$

$$\sigma_{\xi_2\xi_2} = -P_0 - \delta P_1 - \delta^2 P_2 + 2\delta^2\Big(\frac{1}{h_2}\frac{\partial}{\partial \xi_2}\Big(-\frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1}\Big) + \frac{u_1}{h_1h_2}\frac{\partial h_2}{\partial \xi_1}\Big) + O(\delta^3) \,.$$

Observing that in the boundary layer $\psi \sim \delta$, $\xi_2 \sim \delta$ and $\xi_1 \sim 1$, we may reduce the above to

$$\sigma_{\xi_2\xi_2} = -P_0 - \delta P_1 + O(\delta^2)$$

and

$$\sigma_{\xi_1\xi_2} = \delta \Psi_{0NN} + O(\delta^2) \,,$$

where $\psi_0 = \delta\Psi_0$ and $(\xi_2 - \xi_C)h_2(\xi_1, \xi_C) = \delta N$. Thence, returning to (7) and (8), and making use of (6), we obtain at $O(\delta)$,

$$\oint_C \{-P_1 n_1 + \Psi_{0NN} n_2\}_{N=0}\, ds = 0 \,, \tag{9}$$

$$\oint_C \{-P_1 n_2 + \Psi_{0NN} n_1\}_{N=0}\, ds = 0 \,; \tag{10}$$

we note, however, that Eqs (9) and (10) amount to the same condition, since they are different components of the same vector equation. This integral relation forms the basis for calculating the value of the core vorticity, although we must first determine the velocity fields at $O(1)$ and $O(\delta)$ and thence Ψ_{0NN} and P_1 in the boundary layer.

3. The flow at $O(1)$ and $O(\delta)$

Writing the asymptotic expansion for the vorticity, ω, in the core of S as

$$\omega = \omega_0 + \delta\omega_1 + \cdots \,,$$

it is well known from the Prandtl–Batchelor theorem that ω_0 is constant, and that ψ_0 satisfies

$$\nabla^2\psi_0 = -\omega_0$$

subject to boundary condition (3), which is now given by $\psi_0 = 0$ on C. ψ_0 will not, of course, satisfy (4), so a boundary layer of thickness δ, in which $\psi_0 \sim \delta$ and $\omega_0 \sim 1/\delta$, is required. The appropriate equations here, on writing $\omega_0 = \Omega_0/\delta$, are

$$\Psi_{0NN} = -\Omega_0 , \tag{11}$$

$$\Psi_{0N}\Omega_{0s} - \Psi_{0s}\Omega_{0N} = \Omega_{0NN} , \tag{12}$$

subject to

$$\Psi_0 = 0 , \quad \Psi_{0N} = U_s \quad \text{on } N = 0 , \tag{13}$$

$$\Psi_{0N} \to \frac{1}{h_2(s, \xi_C)} \psi_{0_{\xi_2}} , \quad \Omega_0 \to 0 , \quad \text{as } N \to \infty ; \tag{14}$$

in addition, if we denote by L the length of the curve C, then there is a periodicity requirement given by

$$\Psi_0(s + L) = \Psi_0(s) , \tag{15}$$

where s is the arc length which is related to ξ_1 by $s = \int_0^{\xi_1} h_1(x, \xi_C)\,\mathrm{d}x$. The solution of these equations will determine the leading order flow field in the whole of S, up to the constant ω_0.

Although for the purposes of evaluating (9) and (10) we require the value of P_1 at C, it will prove necessary to determine the velocity field at $O(\delta)$ in the whole of S first. The relevant flow equations at $O(\delta)$ are

$$\nabla^2\psi_1 = -\omega_1 , \tag{16}$$

$$u_{0j}\frac{\partial \omega_1}{\partial x_j} + u_{1j}\frac{\partial \omega_0}{\partial x_j} = \delta^2\nabla^2\omega_1 , \tag{17}$$

with the no-slip boundary condition

$$u_{1i}t_i = 0 \quad \text{on } C , \tag{18}$$

and the normal outflow boundary condition, as a result of matching into the boundary layer at $O(\delta)$,

$$\psi_1 = -\int_0^\infty (\Psi_{0N}(s, \infty) - \Psi_{0N}(s, N))\,\mathrm{d}N \quad \text{on } C . \tag{19}$$

In the core of S, (17) implies

$$\omega_1 = \omega_1(\psi_0) . \tag{20}$$

Next, p_1 in the core of S is obtained by solving

$$u_{0i}\frac{\partial u_{1j}}{\partial x_i}+u_{1i}\frac{\partial u_{0j}}{\partial x_i}=-\frac{\partial p_1}{\partial x_j}\,; \tag{21}$$

by writing this as

$$\frac{\partial(u_{0i}u_{1i})}{\partial x_j}-u_{0i}\left(\frac{\partial u_{1i}}{\partial x_j}-\frac{\partial u_{1j}}{\partial x_i}\right)-u_{1i}\left(\frac{\partial u_{0i}}{\partial x_j}-\frac{\partial u_{0j}}{\partial x_i}\right)=-\frac{\partial p_1}{\partial x_j}\,,$$

we may reduce (21) to

$$\frac{\partial}{\partial x_j}\left(u_{0i}u_{1i}+\omega_0\psi_1+\int_0^{\psi_0}\omega_1(s)\,\mathrm{d}s\right)=-\frac{\partial p_1}{\partial x_j}\,,$$

whence

$$p_1=-\left(u_{0i}u_{1i}+\omega_0\psi_1+\int_0^{\psi_0}\omega_1(s)\,\mathrm{d}s\right). \tag{22}$$

In the boundary layer at C, the continuity equation at $O(\delta)$ is

$$\frac{\partial U_1}{\partial s}+\frac{\partial V_1}{\partial N}=-\kappa\left(N\frac{\partial U_0}{\partial s}-V_0\right), \tag{23}$$

with $\kappa(<0)$ as the curvature of C, and the components of the momentum equation are

$$U_0\frac{\partial U_1}{\partial s}+U_1\frac{\partial U_0}{\partial s}+V_0\frac{\partial U_1}{\partial N}+V_1\frac{\partial U_0}{\partial N}+\frac{\partial P_1}{\partial s}-\frac{\partial^2 U_1}{\partial N^2}=$$
$$-\kappa\left(N\frac{\partial^2 U_0}{\partial N^2}+\frac{\partial U_0}{\partial N}-NV_0\frac{\partial U_0}{\partial N}-U_0V_0\right), \tag{24}$$

and

$$\frac{\partial P_1}{\partial N}=-\kappa U_0^2\,. \tag{25}$$

In addition, the boundary conditions are

$$U_1=0,\ V_1=0 \quad \text{on } N=0\,; \tag{26}$$

$$U_1\to u_1(s,\xi_C)+(\omega_0+\kappa u_0(s,\xi_C))N \quad \text{as } N\to\infty\,; \tag{27}$$

$$P_1\to p_1(s,\xi_C)-\kappa u_0^2(s,\xi_C)N \quad \text{as } N\to\infty\,. \tag{28}$$

In particular, (27) and (28) may be obtained by matching to the $O(\delta)$ core flow; the derivation of Eqs (23)–(28) may be found in Schlichting (pp. 144–146) [10] and Van Dyke [12], albeit with a change of sign to take account of the negativity of the curvature which arises because the centre of curvature in the current problem is on the same side of C as the flow. Fortunately, since we are only interested in determining P_1, we do not require to compute a solution to (23)–(28), but may simply write down a formal expression for $P_1(s,0)$, using (22), (25) and (28), as

$$P_1(s,0) = -u_0(s,\xi_C)u_1(s,\xi_C) - \omega_0\psi_1(s,\xi_C) - \kappa \int_0^\infty (u_0^2(s,\xi_C) - U_0^2(s,N))\,\mathrm{d}N\,, \tag{29}$$

(remembering that $\psi_0 = 0$ on C, and that $\psi_1(s,\xi_C)$ is given by (19)).

At this stage, there remains the outstanding problem of determining the core velocity field at $O(\delta)$, for the purposes of evaluating $u_1(s,\xi_C)$ in (29); a simultaneous difficulty is that of determining the function ω_1, which has arisen in the course of our analysis. In order to attempt to do this, we consider the line integral, around any streamline (C_0, say) of the flow field $(u_{0i})_{i=1,2}$, of the primitive-variable equivalent of (17), that is

$$\oint_{C_0} \left(u_{0i}\frac{\partial u_{1j}}{\partial x_i} + u_{1i}\frac{\partial u_{0j}}{\partial x_i} \right) t_j\,\mathrm{d}s = \oint_{C_0} \left(-\frac{\partial p_1}{\partial x_j} + \delta^2 \frac{\partial^2 u_{1j}}{\partial x_i\,\partial x_i} \right) t_j\,\mathrm{d}s\,; \tag{30}$$

using Stokes' theorem, the identity

$$\begin{aligned}\frac{\partial(u_{0i}u_{1i})}{\partial x_j} &= u_{0i}\frac{\partial u_{1j}}{\partial x_i} + u_{1i}\frac{\partial u_{0j}}{\partial x_i} \\ &\quad + u_{0i}\left(\frac{\partial u_{1i}}{\partial x_j} - \frac{\partial u_{1j}}{\partial x_i}\right) + u_{1i}\left(\frac{\partial u_{0i}}{\partial x_j} - \frac{\partial u_{0j}}{\partial x_i}\right),\end{aligned}$$

and the incompressibility condition

$$\frac{\partial u_{1i}}{\partial x_i} = 0\,,$$

(30) may be reduced to

$$\oint_{C_0} (\omega_1 u_{0i}n_i + \omega_0 u_{1i}n_i)\,\mathrm{d}s = \delta^2 \oint_{C_0} \frac{\partial^2 u_{1j}}{\partial x_i\,\partial x_i} t_j\,\mathrm{d}s\,. \tag{31}$$

Now since C_0 is a streamline, the first integral on the left-hand side vanishes. For the second, in the limit as $\delta \to 0$, C_0 becomes a streamline of the core flow, for which ω_0 is constant; this may be taken outside the integral sign, leaving just

$$\omega_0 \oint_{C_0} u_{1i}n_i\,\mathrm{d}s\,,$$

which also vanishes, on using the divergence theorem and the incompressibility condition; in addition, ω_1 tends to the core profile given by Eq. (20), so that

$$\oint_{C_0} \frac{\partial^2 u_{1j}}{\partial x_i\,\partial x_i} t_j\,\mathrm{d}s \to \omega_1'(\psi_0) \oint_{C_0} u_{0i}t_i\,\mathrm{d}s\,, \quad \text{as } \delta \to 0\,.$$

Thus, in the limit as $\delta \to 0$, Eq. (31) is satisfied regardless of the nature of the profile ω_1; however, what is noteworthy is the appearance of $\omega_1'(\psi_0)$ in the last integral. Prompted by its appearance, we assume, not inconsistently with Eq. (31), that $\omega_1'(\psi_0) = 0$, so that ω_1 is constant; although this assumption may seem rather arbitrary, it does enable us to make further analytical progress, which subsequently leads to the satisfactory results obtained in Section 4. We will return to this point again in Section 5.

With ω_1 now taken to be constant, we return to Eqs (16) and (19). Consider the decomposition $\psi_1 = \psi_1^* + \hat{\psi}_1$, where ψ_1^* satisfies

$$\nabla^2 \psi_1^* = -\omega_1 ,$$

subject to $\psi_1^* = 0$ on C, and $\hat{\psi}_1$ satisfies

$$\nabla^2 \hat{\psi}_1 = 0 , \tag{32}$$

subject to

$$\hat{\psi}_1 = -\int_0^\infty (\Psi_{0N}(s, \infty) - \Psi_{0N}(s, N))\, \mathrm{d}N \quad \text{on } C . \tag{33}$$

It is clear that ψ_0 and ψ_1^* satisfy the same equations upto a constant, and hence that we need only look for a canonical solution $\tilde{\psi}$, with $\psi_0 = \omega_0 \tilde{\psi}$ $(u_{0i} = \omega_0 \tilde{u}_i)$, $\psi_1^* = \omega_1 \tilde{\psi}$, $(u_{1i}^* = \omega_0 \tilde{u}_i)$ which satisfies

$$\nabla^2 \tilde{\psi} = -1 , \tag{34}$$

subject to

$$\tilde{\psi} = 0 \quad \text{on } C . \tag{35}$$

In addition, we observe that $\hat{\psi}_1$ is independent of ω_1, and so the only part of (9) and (10), via (29), which depends on ω_1 is

$$-\omega_0 \omega_1 \oint_C (\tilde{u}_j t_j)^2 n_i \, \mathrm{d}s , \quad i = 1, 2 .$$

Now, since $\tilde{u}_j = \tilde{u}_i n_i n_j + \tilde{u}_i t_i t_j$, we may reduce the above expression, on using the condition for zero normal flow, $\tilde{u}_j n_j = 0$, to

$$-\omega_0 \omega_1 \oint_C \tilde{u}_j \tilde{u}_j n_i \, \mathrm{d}s , \quad i = 1, 2 .$$

Applying the divergence theorem, we obtain

$$\oint_C \tilde{u}_j \tilde{u}_j n_i \, \mathrm{d}s = 2 \int\int_S \tilde{u}_j \frac{\partial \tilde{u}_j}{\partial x_i} \, \mathrm{d}A \tag{36}$$

Remembering that (34) can be rewritten as

$$\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} = -1 , \quad i = 1, j = 2 ,$$

and that $(\tilde{u}_i)_{i=1,2}$ satisfies the usual incompressibility condition and the Euler equation, that is

$$\frac{\partial \tilde{u}_j}{\partial x_j} = 0\,,$$

$$\tilde{u}_j \frac{\partial \tilde{u}_j}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i}\,, \quad i = 1, 2\,,$$

where $\tilde{p}$ is the canonical pressure given by $p_0 = \omega_0^2 \tilde{p}$ (and of course $p_1^* = \omega_1^2 \tilde{p}$), we may rewrite (36) as

$$\oint_{C_0} \tilde{u}_j \tilde{u}_j n_i \, \mathrm{d}s = -2 \int\!\!\int_S \left(\frac{\partial(\tilde{p} + \tilde{\psi})}{\partial x_i} \right) \mathrm{d}A\,. \tag{37}$$

Thence we obtain, on applying the divergence theorem,

$$\begin{aligned} \int\!\!\int_S \left(\frac{\partial(\tilde{p} + \tilde{\psi})}{\partial x_i} \right) \mathrm{d}A &= -\oint_C (\tilde{\psi} + \tilde{p}) n_i \, \mathrm{d}s \\ &= 0\,, \end{aligned} \tag{38}$$

by virtue of (35) and (6). In summary, therefore, ω_0 may be determined using either Eqs (9) or (10), in which $(\Psi_{0NN})_{N=0}$ is independent of ω_1 and in which the part of the integral of $(P_1)_{N=0}$ which involves ω_1 vanishes; hence, ω_0 is independent of ω_1.

We now proceed to demonstrate how to calculate ω_0 for a specific example.

4. Example

We take the bounding curve C to be an ellipse with semi-major and -minor axes a and b respectively, so that C is given, in Cartesian coordinates, by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

It is appropriate here to introduce elliptic coordinates (ξ, η), which are related to (x, y) coordinates by

$$x = 2\,\mathrm{e}^{-\eta_0} \cosh \eta \cos \xi\,, \qquad y = 2\,\mathrm{e}^{-\eta_0} \sinh \eta \sin \xi\,,$$

where $\eta = \eta_0$ is the ellipse C so that

$$a = 2\,\mathrm{e}^{-\eta_0} \cosh \eta_0\,, \qquad b = 2\,\mathrm{e}^{-\eta_0} \sinh \eta_0\,,$$

and the eccentricity e of the ellipse is given by

$$e = \left\{ 1 - \left(\frac{b}{a} \right)^2 \right\}^{1/2} = \operatorname{sech} \eta_0\,.$$

The canonical solution $\tilde{\psi}$, and hence ψ_0 and ψ_1^*, is determined by the solution of (34), subject to $\tilde{\psi} = 0$ on C. In Cartesian coordinates, this is easily found to be

$$\tilde{\psi} = -\frac{a^2 b^2}{2(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),$$

rewritten in elliptic coordinates as

$$\tilde{\psi} = -\frac{2\,\mathrm{e}^{-2\eta_0}\cosh^2\eta_0\sinh^2\eta_0}{\cosh^2\eta_0+\sinh^2\eta_0}\left(\frac{\cosh^2\eta\cos^2\xi}{\cosh^2\eta_0}+\frac{\sinh^2\eta\sin^2\xi}{\sinh^2\eta_0}-1\right).$$

Thence, we derive the inviscid tangential velocity at the edge of the boundary layer to be

$$u_0(s,\eta_0) = \frac{1}{h_2(\xi,\eta_0)}\left(\frac{\partial\psi_0}{\partial\eta}\right)_{\eta=\eta_0} = \omega_0\,\mathrm{e}^{-\eta_0}\tanh 2\eta_0(\sinh^2\eta_0+\sin^2\xi)^{1/2},$$

where

$$h_1(\xi,\eta) = h_2(\xi,\eta) = 2\,\mathrm{e}^{-\eta_0}(\sinh^2\eta+\sin^2\xi)^{1/2},$$

and s and ξ are related by $s=\int_0^\xi (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}\,\mathrm{d}\theta$, so that the length ($L$) of C, is given by $L=\int_0^{2\pi}(a^2\sin^2\theta+b^2\cos^2\theta)^{1/2}\,\mathrm{d}\theta$. Lastly, in order to close the problem posed by Eqs (11)–(15), we prescribe

$$U_s = 1 + \frac{3}{4}\cos\xi\,; \tag{39}$$

this choice is entirely arbitrary, but it does enable us to make a direct comparison with earlier results [9].

In order to compute a numerical solution to the leading order boundary-layer equations, the well-known Keller Box method [3] was used. However, instead of discretising (11) and (12), it is much easier to combine these equations and to discretise the resulting first integral with respect to N, that is

$$\Psi_{0NNN} + u_0(s,\eta_0)\frac{\mathrm{d}u_0(s,\eta_0)}{\mathrm{d}s} = \Psi_{0N}\Psi_{0Ns} - \Psi_{0NN}\Psi_{0s}\,;$$

Eqs (13)–(15) remain, except that the second half of (14) is redundant. In order to start the integration, the initial profile was taken to be

$$\Psi_0 = U_s N + (u_0(s,\eta_0) - U_s)(N + \mathrm{e}^{-N} - 1)\,;$$

this has the property that it satisfies the boundary conditions at $s=0$. For a given value of ω_0, the integration was carried out for as many periods as was necessary to satisfy the periodicity condition (15); after k periods (say), the value of $|\Psi_0(kL,N)-\Psi_0((k-1)L,N)|$ was checked at each mesh point, and the integration was stopped, and a converged solution deemed to have been obtained, if $\max_{N\in[0,N_\infty]}|\Psi_0(kL,N)-\Psi_0((k-1)L,N)|$ was less than some prescribed tolerance.

After a converged solution has been obtained, it is then necessary to solve the Dirichlet problem, given by (32) and (33), for $\hat{\psi}_1$; the right-hand side of (33) is obtained from the solution of the leading order boundary-layer equations. This is done most simply in (ξ,η) coordinates, and the solution is found to be

$$\hat{\psi}_1 = A\eta + \sum_{n=1}^{\infty} (B_n e^{n\eta} + C_n e^{-n\eta}) \cos n\xi + \sum_{n=1}^{\infty} D_n \sinh n\eta \sin n\xi \ , \tag{40}$$

where

$$D_n = \frac{1}{\sinh n\eta_0} \int_0^{2\pi} f(\xi) \sin n\xi \, d\xi \ ,$$

and A, B_n and C_n are related by

$$A\eta_0 + B_0 + C_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \, d\xi \ ,$$

$$B_n e^{n\eta_0} + C_n e^{-n\eta_0} = \frac{1}{\pi} \int_0^{2\pi} f(\xi) \cos n\xi \, d\xi \ , \quad n = 1, 2, \ldots ,$$

where $f(\xi) = -\int_0^\infty (\Psi_{0N}(s, \infty) - \Psi_{0N}(s, N)) \, dN$. Although A, B_n and C_n remain undetermined, this does not actually matter as far as finding ω_0 is concerned, although we return to this point in Section 5.

Defining

$$\hat{u}_1(\xi, \eta_0) := \frac{1}{h_2(\xi, \eta_0)} \left(\frac{\partial \hat{\psi}_1}{\partial \eta} \right)_{\eta = \eta_0} , \tag{41}$$

we now require to evaluate, for the purposes of satisfying (10), the integrals

$$I := a \int_0^{2\pi} \Big[-u_0(\xi, \eta_0)\hat{u}_1(\xi, \eta_0) - \omega_0 f(\xi) - \kappa(\xi) \int_0^\infty (u_0^2(\xi, \eta_0) - U_0^2(\xi, N)) \, dN \Big] \sin \xi \, d\xi \ ,$$

$$J := b \int_0^{2\pi} \Psi_{0NN} \cos \xi \, d\xi \ ,$$

where

$$\kappa(\xi) = -\frac{(a^2 \sin^2 \xi + b^2 \cos^2 \xi)^{3/2}}{ab} \ .$$

Using (40) and (41), we may simplify the first of these to obtain

$$I = aD_1 \omega_0 \pi \left(\frac{\tanh 2\eta_0}{2 \tanh \eta_0} - 1 \right) - a \int_0^{2\pi} \kappa(\xi) \left(\int_0^\infty (u_0^2(\xi, \eta_0) - U_0^2(\xi, N)) \, dN \right) \sin \xi \, d\xi \ .$$

I and J were subsequently calculated for different values of ω_0 using NAG Routine D01GAF, and a straightforward bisection technique was used in order to determine the value of ω_0 for which $I = J$. Initial guesses for ω_0 were taken to be $\omega_{01} = 2.25$, $\omega_{02} = 2.35$, and bisection was carried out until $|I - J| < 10^{-7}$, a procedure which required around 20 iterations. To begin with, trial runs were carried out for the case of a circular boundary

Table 1. Results for circular boundary

Δs	ΔN	N_∞	ω_0
$\frac{\pi}{50}$	0.1	10	2.230
$\frac{\pi}{25}$	0.05	10	2.221
$\frac{\pi}{25}$	0.01	20	2.232
$\frac{\pi}{50}$	0.01	20	2.257
$\frac{\pi}{50}$	0.05	20	2.256

(formally, the limit as $\eta_0 \to \infty$); from [1], the core vorticity can be determined using the relation

$$\int_0^{2\pi} U_s^2 \, d\xi = \int_0^{2\pi} u_0^2(\xi, \infty) \, d\xi ,$$

which, with U_s as in (39) and $u_0(\xi, \infty) = \frac{1}{2}\omega_0$, leads to

$$\omega_0 = \sqrt{\frac{41}{8}} \approx 2.263846 .$$

This result provided the basis for testing the reliability of the method; computations were carried out for a range of mesh sizes (with Δs denoting the mesh spacing in s and $\Delta\eta$ denoting the mesh spacing in η) and two values for the location (N_∞) of the outer edge of the boundary layer, the results for which are shown in Table 1.

These indicate that the combination of a coarse mesh and small value of N_∞ gives values of ω_0 which are significantly lower than the true value, although the case for which $N_\infty = 20$, $\Delta s = \pi/50$, $\Delta\eta = 0.01$ gives a much better result, which might be improved upon by taking an even finer mesh and a larger value of N_∞. Subsequently, the meshes with $N_\infty = 20$ were adopted for the numerical integration of the boundary-layer equations for an elliptic boundary, and Richardson extrapolation was used to determine the value of ω_0; for the circular boundary, this procedure gave $\omega_0 = 2.265$.

The results obtained are summarised in Figs 3 and 4. Figure 3 is a plot of the core vorticity ω_0 against the ellipse eccentricity e; it was not possible to obtain a solution for values of e exceeding 0.8, a feature that will be discussed in the next section. On the same plot, we include for comparison the results from [9]; these provide good agreement with our results for the dependence of ω_0 on e. In addition, Fig. 4 shows the dependence of the integral I (and hence J) on e.

5. Conclusions

In this paper, we have presented an asymptotic method for calculating the uniform vorticity (ω_0) in a high Reynolds number flow with closed streamlines. Asymptotic analysis of the

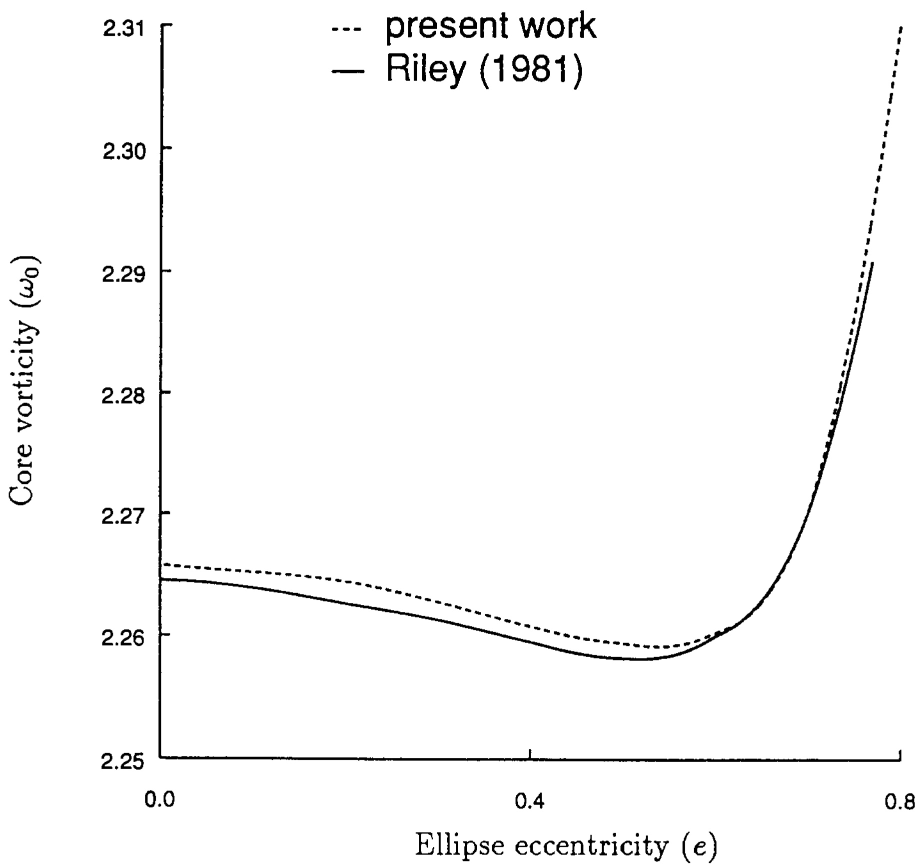

Fig. 3. Plot of core vorticity (ω_0) vs. ellipse eccentricity (e).

Navier–Stokes equations led to an integral relation containing ω_0 implicitly; the integral could be evaluated only by first considering the velocity fields at $O(1)$ and $O(\delta)$. Subsequently, this was done for the case of a family of elliptical boundaries and the particular case of a circular boundary. The method yielded good agreement with the analytic result that exists for the case of a circular boundary in [1], and with the numerical results for a family of ellipses obtained by Riley [9]. However, several other points arise from the present work.

First, the fact that a solution to the boundary-layer equations could not be obtained for values of e in excess of 0.8 is partially consistent with [9]; numerically, the equations display a behaviour similar to that encountered in boundary-layer separation, namely that the tangential velocity Ψ_{0N} becomes negative at some point in the boundary layer. Riley [9], however, was unable to obtain solutions for values of e greater than 0.77, although the slight discrepancy is perhaps unsurprising, as Riley's criterion for determining ω_0 is different to the one used here. A second observation is that the method presented here does not produce the value of ω_0 that is obtained by solving the full Navier–Stokes equations numerically, as in [5]. This does not, however, invalidate the current results, since there is doubt in [5] as to whether the values for Re used there were large in an asymptotic sense, even though they were at the upper limit for which a converged solution to the Navier–Stokes equations could be obtained. Thirdly, we recall that we did not appear to use (9) in determining ω_0 in Section

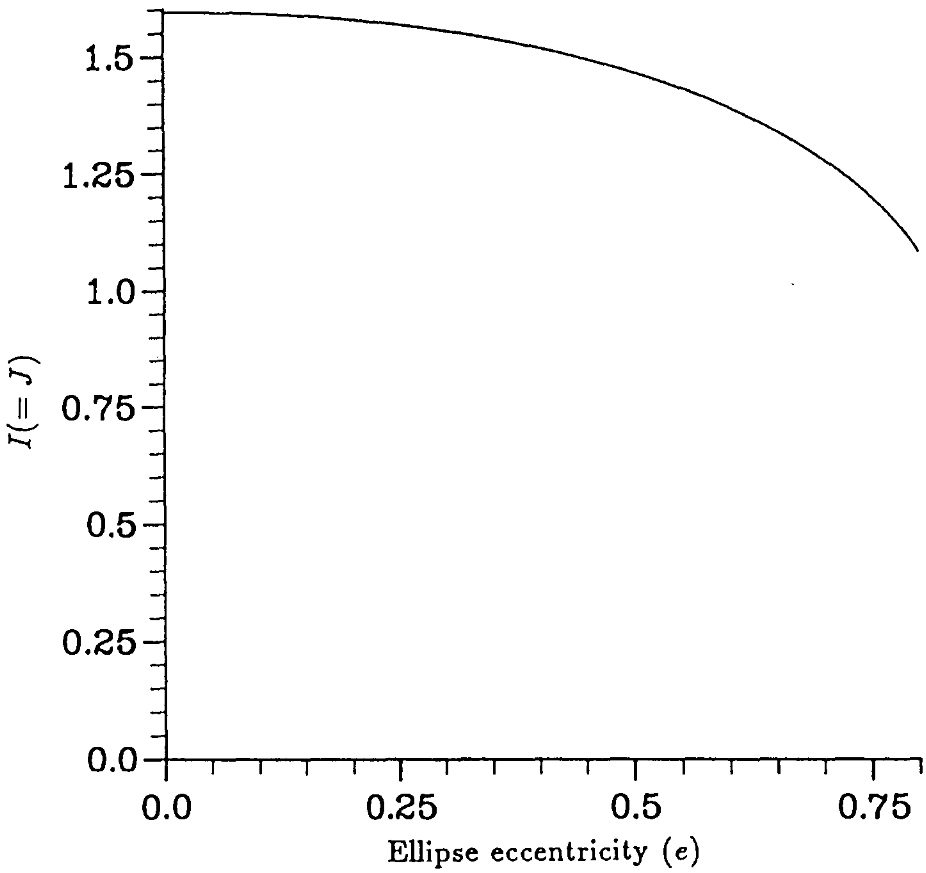

Fig. 4. Plot of integral, I, vs. ellipse eccentricity (e) for $\Delta\eta = 0.05$, $\Delta s = \pi/50$.

4. As was stated in Section 2, (9) and (10) are the two components of the first integral of the Navier–Stokes equations, and hence amount to the same condition; however, to determine the various terms in (9) would require us to evaluate the coefficients A, B_n and C_n in (40). This cannot be done by considering the Dirichlet problem in elliptic coordinates, and one would either have to find a conformal map from the interior of the ellipse to the interior or exterior of a circle and solve the resulting Dirichlet problem, or resort to a numerical solution; these tasks have been omitted here in view of the fact that to use (10) requires much less effort.

Fourthly, we remark on the computing time that was required for the integrations. Given the small values of $\Delta\eta$ and Δs that were used, the fact that boundary-layer equations were periodic and that the root for ω_0 was found using the bisection method, it is clear that the calculations were quite lengthy; on average, between 30 and 40 periods were required to satisfy the periodicity condition, which was taken numerically to be

$$\max_{N\in[0,N_\infty]} |\Psi_0(kL, N) - \Psi_0((k-1)L, N)| < \epsilon$$

with k an integer, and $\epsilon = 0.0001$. To obtain a converged solution for a particular value of ω_0 and e, approximately two hours of CPU time on a Sparc 2 Workstation were required.

Fifth, we must note that although the analysis of Sections 2 and 3 appears to be valid for any laminar high Reynolds number flow with closed streamlines, it is clear that in practice this may not be necessarily so, since the method relies on the solution of the viscous boundary-layer equations. The ellipses of eccentricity greater than 0.8 in Section 4 provide one example where the method breaks down; in addition, it is likely that this situation will also occur in flows containing inviscid stagnation points, such as the classical driven-cavity problem.

Lastly, we return to the assumption made in Section 3 that the core vorticity at $O(\delta)$ should be constant. With hindsight, this assumption proved to have been a favourable one to have made: it enable further reduction of the equations, leading to the conclusion, perhaps surprising in view of the algebraic manipulation involved, that the $O(1)$ core vorticity did not explicitly depend on the $O(\delta)$ flow field; furthermore, the results obtained in Section 4 using this assumption compared very favourably with those of Riley [9]. Given the above, it may seem plausible therefore to suggest that the $O(\delta)$ core vorticity should be constant for high Reynolds number flows within arbitrarily-shaped closed boundaries, although it is clear that further work to validate this conjecture, such as the full solution of the Navier–Stokes equations at $O(\delta)$ for a wide set of geometries, will be necessary.

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